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LETTER TO THE EDITOR

Anomalous decay of pair correlations for two-dimensional critical wetting

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Abstract. We develop a scaling theory for the decay of the spin–spin correlation function at two-dimensional ($d=2$) wetting transitions in systems with short-ranged forces. For the critical wetting transition effective Hamiltonian results show that the scaling theory is obeyed but the decay of correlations is anomalous. In contrast, for the complete wetting transition, scaling and Ornstein–Zernike theory are both valid. We argue that the anomalous decay is specific to zero bulk field critical wetting transitions and $d=2$.

The Wu anomaly (Wu 1966) refers to the anomalous asymptotic decay of the connected spin–spin correlation function $G(\boldsymbol{r})$ in the zero field ($H=0$), subcritical ($T<T_c$) two-dimensional ($d=2$) Ising model. Recall that according to the classical Ornstein–Zernike (OZ) theory (Ornstein and Zernike 1918) $G(\boldsymbol{r})$ should decay as

$$G(\boldsymbol{r}) \sim e^{-r/\xi(\theta,T)}/r^{(d-1)/2} \qquad \frac{r}{\xi(\theta,T)} \to \infty \tag{1}$$

for spin separations $r \equiv |\boldsymbol{r}|$ much greater than the correlation length $\xi(\theta, T)$. Here θ refers to the angle of the spin separation vector $\boldsymbol{r}$ WRT an arbitrary axis. However exact analysis (Wu 1966, McCoy and Wu 1973) reveals an anomalous asymptotic decay law for $T<T_c$ and $H=0$:

$$G(\boldsymbol{r}) \sim e^{-2r/\xi(\theta,T)}/r^2. \tag{2}$$

The exponent of the power in the decay law is 2 rather than the OZ value $\frac{1}{2}$. This breakdown of OZ theory for the non-critical bulk correlation function is believed to be specific to $d=2$, $H=0$ and $T<T_c$ (Fisher and Camp 1971) and can be quantitatively understood in terms of the solid-on-solid/bubble model of Abraham (1983) or equivalently the random walk picture of Fisher (1984). Here we point out that similar anomalous decay is found in two dimensions at the strong-fluctuation regime critical wetting phase transition (for a review see e.g. Dietrich 1988).

Consider a semi-infinite Ising plane with bulk field H and surface field H_1. Let J_1 and J_2 denote the usual nearest-neighbour exchange interactions in the z (normal to

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the surface) and x (parallel) directions respectively. For any (fixed) $T < T_c$ the system exhibits a *critical wetting* transition when $H_1 = H_1^w$ (>0 say) satisfies (Abraham 1980)

$$e^{2J_2/k_BT}\left(\cosh\frac{2J_1}{k_BT} - \cosh\frac{2H_1}{k_BT}\right) = \sinh\frac{2J_1}{k_BT} \tag{3}$$

with $H = 0^-$. As $t \equiv (H_1^w - H_1)/H_1^w \to 0^+$ the thickness l of an adsorbed layer of upspins diverges continuously, i.e. $l \sim t^{-\beta_s}$ with $\beta_s = 1$. Associated with the divergence of l is the growth of large correlation lengths along ($\xi_\parallel \sim t^{-\nu_\parallel}$ with $\nu_\parallel = 2$) and normal ($\xi_\perp \sim t^{-\nu_\perp}$ with $\nu_\perp = 1$) to the surface. Effective interfacial Hamiltonian models (see e.g. Burkhardt 1981) correctly describe this critical behaviour. Moreover the universality hypothesis implies that the scaling properties of one- and two-point functions are the same for both the Ising and effective Hamiltonian models. This is known to be true for the energy density and energy–energy correlation function (Ko and Abraham 1989, Burkhardt 1989). For $H_1 > H_1^w$ (with T fixed) the *complete wetting* phase transition corresponds to the divergence of l, $\xi_\perp$ and $\xi_\parallel$ as $H \to 0^-$. Although the full Ising model has not been solved for this transition solid-on-solid (SOS) and continuum capillary-wave Hamiltonian studies (Abraham and Smith 1982, Lipowsky 1985) yield the analogous critical behaviour $l \sim |H|^{-\beta_s^{CO}}$ (with $\beta_s^{CO} = \frac{1}{3}$), $\xi_\perp \sim |H|^{-\nu_\perp^{CO}}$ (with $\nu_\perp^{CO} = \frac{1}{3}$) and $\xi_\parallel \sim |H|^{-\nu_\parallel^{CO}}$ (with $\nu_\parallel^{CO} = \frac{2}{3}$). The values of these critical exponents for $d = 2$ critical and complete wetting as well as other quantities of interest, such as the shape of adsorbed droplets can be understood using random walk arguments (Fisher 1984). As stated earlier SOS model Hamiltonians and random walk approaches can also be used to explain the anomalous decay of $G(\boldsymbol{r})$ in the bulk. This is suggestive that effective Hamiltonian models of two-dimensional wetting transitions may also reveal analogous anomalous decay.

To place our results in context we develop a simple scaling theory of the form of G at wetting transitions. For simplicity here we largely confine our analysis to short-ranged forces in two dimensions, although it is easy to extend the scaling theory to account for long-ranged forces and arbitrary dimensionality. The critical and complete wetting transitions of the planar Ising model described above belong to the ($d = 2$) strong-fluctuation (SFL) and weak-fluctuation (WFL) scaling regimes, respectively. These scaling regimes characterize, quite generally, wetting transitions for d below the upper critical dimension $d_>$ (Lipowsky and Fisher 1987). In these fluctuation dominated regimes, hyperscaling and critical exponent relations suggest that $l \sim \xi_\perp$ so that the interface depins and delocalizes with the same critical exponent (Kroll *et al* 1985). On this basis we have argued (Parry 1991a, b) that the transverse moments of G are scaled functions. It is easy to extend this theory to G itself. We suppose that the spins are at distances z_1, z_2 from the wall respectively and are separated by the parallel displacement $x_{12} = x_1 - x_2$. Following the Weeks (1984) scaling argument for interfacial delocalization in a gravitational field, we assume that in the SFL and WFL regimes distances can only appear in scaling form. For the $d = 2$ Ising model critical wetting transition we write the connected spin–spin correlation function as $G = G(z_1, z_2; x_{12})$ and postulate that G contains a singular scaling contribution

$$G^{\text{sing}}(z_1, z_2; x_{12}) = G_{\text{SC}}^{-}(z_1 t^{\nu_\perp}, z_2 t^{\nu_\perp}; x_{12} t^{\nu_\parallel}) \tag{4}$$

for $H_1 \leqslant H_1^w$ and $h \equiv |H|/k_BT = 0$. If $h \neq 0$ then we must allow for an extra scaling variable $ht^{-\Delta}$ in the RHS of (4). Δ is the gap exponent ($\Delta = 3$ in $d = 2$). For the complete wetting transition the analogous scaling hypothesis is

$$G^{\text{sing}}(z_1, z_2; x_{12}) = G_{\text{SC}}^{+}(z_1 h^{\nu_\perp^{CO}}, z_2 h^{\nu_\perp^{CO}}; x_{12} h^{\nu_\parallel^{CO}}) \tag{5}$$

valid for $h \to 0$ and $|t|h^{-1/\Delta} \to \infty$. This latter condition means that we do not consider the subtle crossover effects associated with the limit $H_1 \to H_1^{w+}$ for $h \neq 0$ (Parry and Evans 1992).

For wetting transitions the exponent analogous to $\eta = 0\,\forall d$ (Lipowsky and Fisher 1987) so it is natural to assume, in addition to the above scaling argument, that the transverse Fourier transform $\tilde{G}(z_1, z_2; Q)$ defined (taking the continuum limit) in $d = 2$ by

$$\tilde{G}(z_1, z_2; Q) \equiv \int_{-\infty}^{+\infty} e^{iQx_{12}} G(z_1, z_2; x_{12})\, dx_{12} \tag{6}$$

contains a simple OZ contribution (Henderson 1991) $\forall z_1 z_2$

$$\tilde{G}_{OZ}^{+,-}(z_1, z_2; Q) = \frac{\tilde{G}_0^{+,-}(z_1, z_2)}{1 + \xi_\parallel^2 Q^2} \qquad \xi_\parallel^2 Q^2 \ll 1 \tag{7}$$

where $\tilde{G}_0^{+-}(z_1, z_2)$ denotes the zeroth moment. In the complex Fourier plane the asymptotic decay of $G_{OZ}(z_1, z_2; x_{12})$ is determined by the simple poles at $Q = \pm i/\xi_\parallel$. Combining (7) with the scaling hypothesis above we find

$$G_{OZ}^{+,-}(z_1, z_2; x_{12}) \sim \Phi^{+,-}\left(\frac{z_1}{\xi_\perp}, \frac{z_2}{\xi_\perp}\right) e^{-x_{12}/\xi_\parallel} \left(\frac{x_{12}}{\xi_\parallel}\right)^{(2-d)/2} \tag{8}$$

for $x_{12}/\xi_\parallel \to \infty$. Note that we have written the dimension dependence of the power law explicitly. The scaling functions $\Phi^{+,-}$ are trivially related to the scaling properties of the zeroth moments $\tilde{G}_0^{+,-}(z_1, z_2) \equiv \tilde{G}^{+,-}(z_1, z_2; 0)$ (Parry 1991a). For SFL critical wetting $\Phi^-(u, v)$ has the short-distance expansion

$$\Phi^-(u, v) \approx (uv)^{2(d-1)/(3-d) - 1/\nu_\perp} \qquad u, v \to 0 \tag{9a}$$

whilst for the WFL complete wetting transition

$$\Phi^+(u, v) \approx (uv)^{(d+1)/(3-d)} \qquad u, v \to 0. \tag{9b}$$

These SDE results should be valid regardless of whether the power law in the correct asymptotic expansion of $G(z_1, z_2; x_{12})$ is OZ-like or not (see later). Below we shall show that the OZ prediction (8) is not obeyed in $d = 2$ for the SFL critical wetting transition.

To calculate $G(z_1, z_2; x_{12})$ for $d = 2$ critical and complete wetting we use the standard continuum effective Hamiltonian

$$H[l(x)] = \int_{-\infty}^{+\infty} dx \left\{ \frac{\Sigma(T)}{2} \left(\frac{dl}{dx}\right)^2 + V(l(x)) \right\} \tag{10}$$

with $\Sigma(T)$ the surface stiffness coefficient and $V(l)$ the binding potential. More generally such Hamiltonians should describe correctly the critical behaviour and scaling properties of interfaces above their roughening temperature ($T > T_R$) and away from bulk criticality. Recall that $T_R = 0$ in the $d = 2$ Ising model. The single-valued graph $l(x) \geq 0$ separates regions of upspin and downspin and models the intrinsic interface that fluctuates near the surface at $z = l = 0$. For critical wetting in $d = 2$ with a contact binding potential and zero bulk field ($h = 0$) $G(z_1, z_2; x_{12})$ can be calculated in closed form (Parry 1991a) using the standard SOS prescription for constructing pair correlations from probability distributions (Burkhardt 1989):

$$G(z_1, z_2; x_{12}) \propto e^{-2z_1/\xi_\perp} \operatorname{erfc}\left[\frac{(z_2 - z_1)}{2\xi_\perp}\sqrt{\frac{\xi_\parallel}{|x_{12}|}} + \sqrt{\frac{|x_{12}|}{\xi_\parallel}}\right]$$
$$+ e^{-2z_2/\xi_\perp} \operatorname{erfc}\left[\frac{(z_1 - z_2)}{2\xi_\perp}\sqrt{\frac{\xi_\parallel}{|x_{12}|}} + \sqrt{\frac{|x_{12}|}{\xi_\parallel}}\right]$$
$$- e^{-2(z_1+z_2)/\xi_\perp} \operatorname{erfc}\left[\frac{-(z_1 + z_2)}{2\xi_\perp}\sqrt{\frac{\xi_\parallel}{|x_{12}|}} + \sqrt{\frac{|x_{12}|}{\xi_\parallel}}\right]$$
$$- \operatorname{erfc}\left[\frac{(z_1 + z_2)}{2\xi_\perp}\sqrt{\frac{\xi_\parallel}{|x_{12}|}} + \sqrt{\frac{|x_{12}|}{\xi_\parallel}}\right]. \tag{11}$$

Equation (11) is of the scaling form (5). For large $x_{12} \gg \xi_\parallel$ (11) implies the asymptotic decay

$$G(z_1, z_2; x_{12}) \approx \left(\frac{z_1 z_2}{\xi_\perp \xi_\perp}\right) e^{-(z_1+z_2)/\xi_\perp} e^{-x_{12}/\xi_\parallel}\left(\frac{x_{12}}{\xi_\parallel}\right)^{-3/2} + \ldots \tag{12}$$

which identifies the correct scaling function $\Phi^-(u, v)$. The SDE of Φ^- is in agreement with (9*a*); recall $\nu_\perp = 1$ in $d = 2$. However the form of the x_{12} decay law (12) is anomalous and not of the OZ type (8). This reflects the fact that $\tilde{G}(z_1, z_2; Q)$ does not have simple poles at $Q = \pm i/\xi_\parallel$. The singularities in $\tilde{G}(z_1, z_2; Q)$ are not isolated which in turn reflects the continuum of scattering state eigenfunctions in the Schrödinger problem (Burkhardt 1981). These subtleties do not affect the scaling and SDE properties of the transverse moments of G (Parry 1991a). These moments still exist away from the transition temperature. Further, their scaling properties follow immediately from the ansatz (4) which does not make any assumption about the form of $\tilde{G}(z_1, z_2; Q)$. If we allow for $h \neq 0$ it is straightforward to show that $G(z_1, z_2; x_{12})$ then has a pure exponential decay for large x_{12} consistent with the OZ prediction in $d = 2$. The anomalous decay is therefore restricted to $h = 0$.

For the case of complete wetting i.e. $H_1 > H_1^{\mathrm{w}}$ and $H \to 0^-$ $G(z_1, z_2; x_{12})$ has not been calculated in closed form but it is straightforward to show, using standard transfer matrix/integral techniques, that $G(z_1, z_2; x_{12})$ has a scaling contribution of the form (8). Moreover the discreteness of the transfer integral spectrum necessarily implies the OZ decay

$$G^+(z_1, z_2; x_{12}) \approx e^{-x_{12}/\xi_\parallel} \Phi^+\left(\frac{z_1}{\xi_\perp}, \frac{z_2}{\xi_\perp}\right) \qquad \frac{x_{12}}{\xi_\parallel} \to \infty \tag{13a}$$

with

$$\Phi^+(u, v) = \int_0^u \int_0^v \operatorname{Ai}(y_1 + \lambda_0)\operatorname{Ai}(y_1 + \lambda_1)\operatorname{Ai}(y_2 + \lambda_0)\operatorname{Ai}(y_2 + \lambda_1)\, dy_1\, dy_2 \tag{13b}$$

where λ_0 and λ_1 are the largest and next largest zeros of the Airy function Ai. Equation (13) has the predicted SDE (9*b*). The OZ decay of correlations at complete wetting is similar to that found for the well studied problem of interfacial delocalization in a gravitational field (Weeks (1984) and references therein).

We conclude that for systems with short-ranged forces in $d = 2$ OZ is valid for complete wetting but invalid for critical wetting provided $H = 0$. In $d = 3$ we do not expect any anomalous decay even in the case of critical wetting with short-ranged forces (Lipowsky *et al* 1983, Brezin *et al* 1983, Fisher and Huse 1985). Using the matching procedure of Fisher and Huse (1985) it follows that it is possible to renormalize the binding potential until a Gaussian approximation to the renormalized Hamiltonian is appropriate. This is a sufficient condition to ensure that $G(z_1, z_2; R)$ (with R the parallel separation of the particles measured along the two dimensional substrate) decays as $e^{-R/\xi_\parallel}(R/\xi_\parallel)^{-1/2}$ for $R \gg \xi_\parallel$. This is precisely the OZ prediction.

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